UNIVERSITY OF TWENTE.

Formal Methods & Tools.

Symbolic reductions of probabilistic models using linear process equations

Mark Timmer

January 18, 2011

FMT Lunchmeeting

Joint work with
Mariëlle Stoelinga and Jaco van de Pol

Contents

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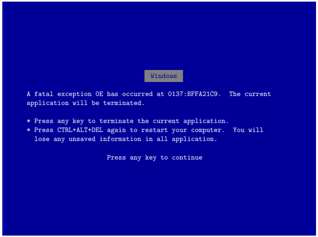
Introduction – Dependability

Dependability of computer systems is becoming more and more important.



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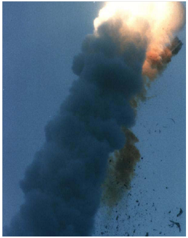

Ariane 5 crash

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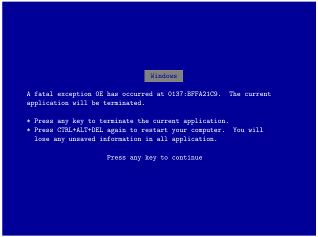

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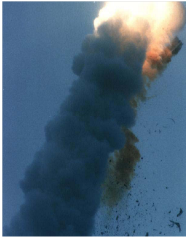
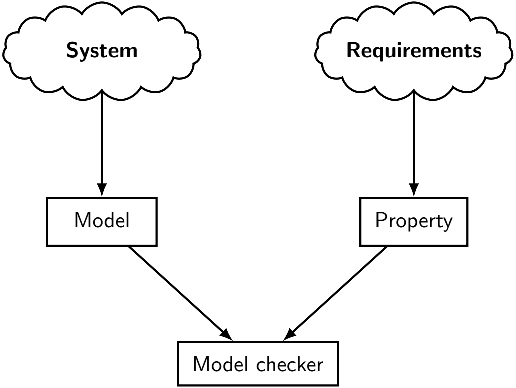


Ariane 5 crash

Our aim: use quantitative formal methods to improve system quality.

Introduction – Model Checking

A popular solution is model checking; verifying properties of a system by constructing a model and ranging over its state space.



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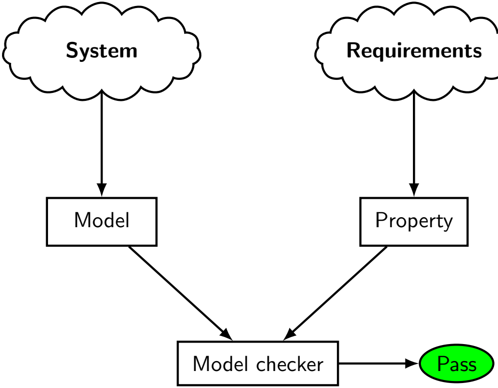

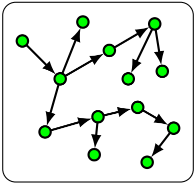

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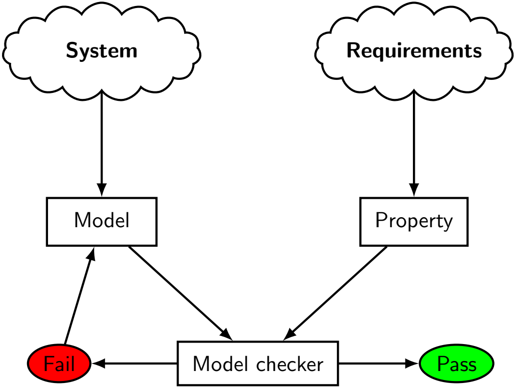

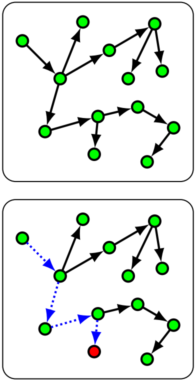

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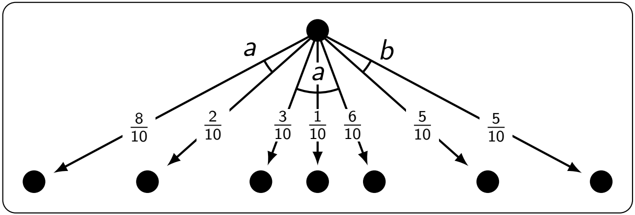

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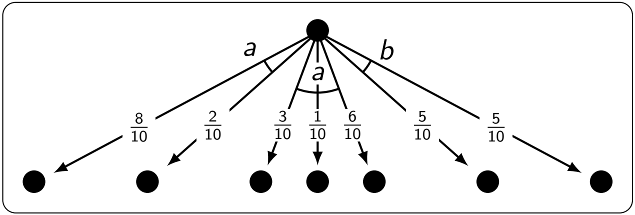

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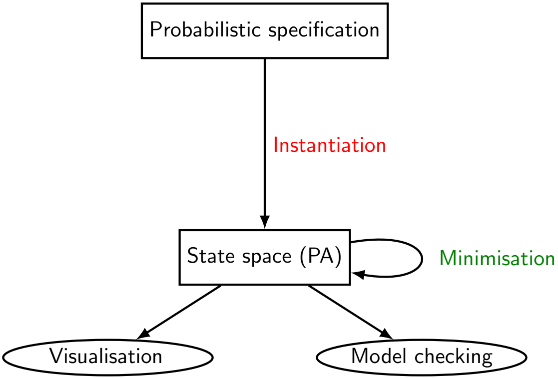
Limitations of previous approaches:

- Susceptible to the state space explosion problem
- Restricted treatment of data

Overview of our approach



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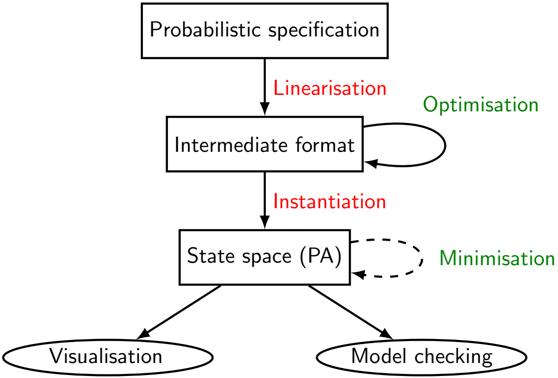

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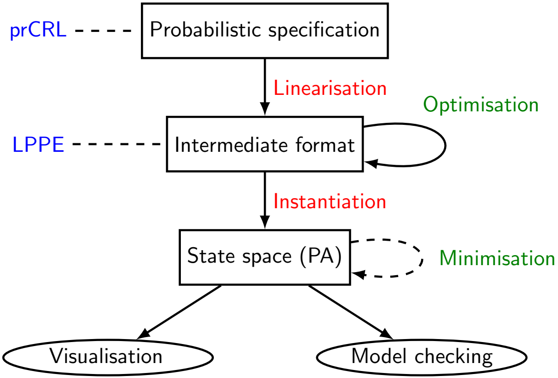

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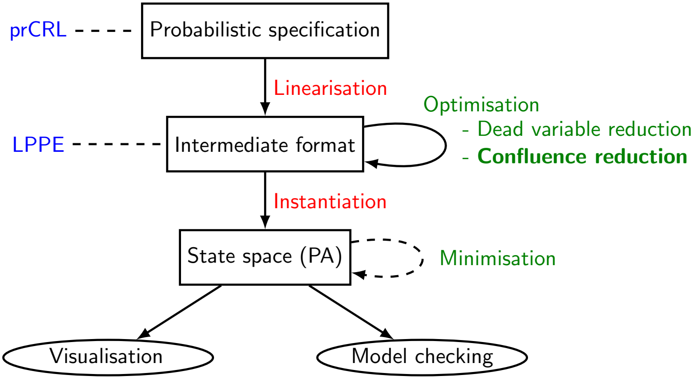
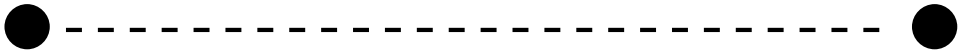


Equivalences: probabilistic bisimulation

Notions of equivalence: strong/branching probabilistic bisimulation



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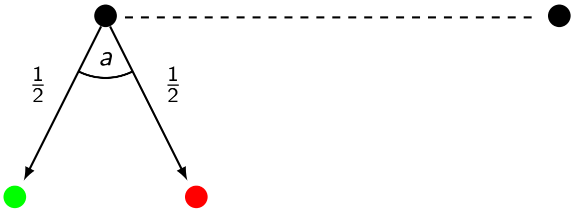

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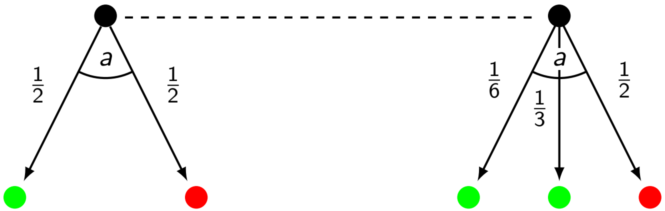

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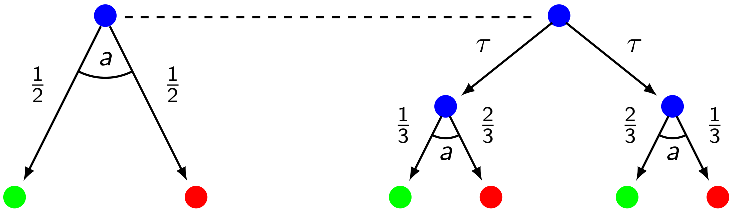

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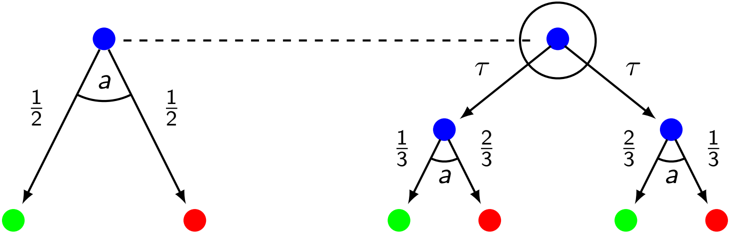

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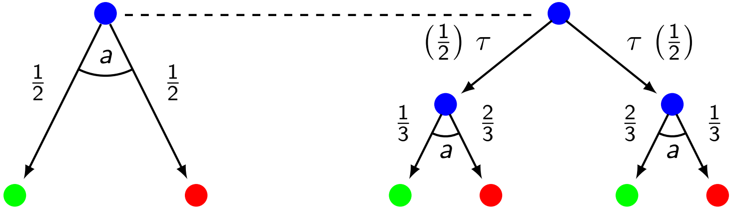

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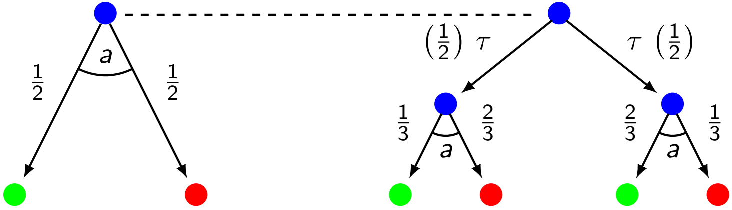

Probability of green: $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$

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A process algebra with data and probability: prCRL

Specification language prCRL:

- Based on μCRL (so data), with additional probabilistic choice
- Semantics defined in terms of probabilistic automata
- Minimal set of operators to facilitate formal manipulation
- Syntactic sugar easily definable

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The grammar of prCRL process terms

Process terms in prCRL are obtained by the following grammar:

$$p ::= Y(\boldsymbol{t}) \mid c \Rightarrow p \mid p + p \mid \sum_{x:D} p \mid a(\boldsymbol{t}) \sum\!\!\!\!\bullet_{x:D} f : p$$

Process equations and processes

A process equation is something of the form $X(\boldsymbol{g} : \boldsymbol{G}) = p$.

An example specification

Sending an arbitrary natural number

$$X(\text{active} : \text{Bool}) = \text{not}(\text{active}) \Rightarrow \text{ping} \cdot \sum_{b:\text{Bool}} X(b) + \text{active} \Rightarrow \tau \sum\nolimits^{\bullet}_{n:\mathbb{N}^{>0}} \frac{1}{2^n} : \Big(\text{send}(n) \cdot X(\text{false})\Big)$$

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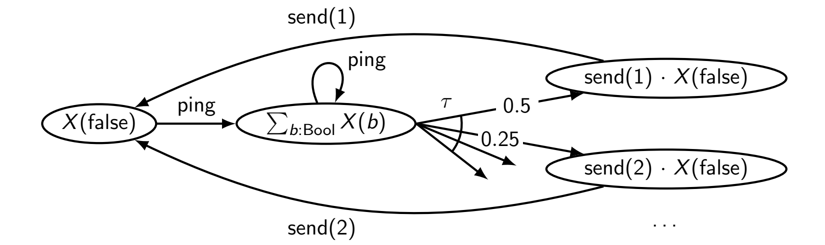

Composability using extended prCRL

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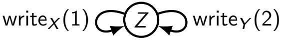

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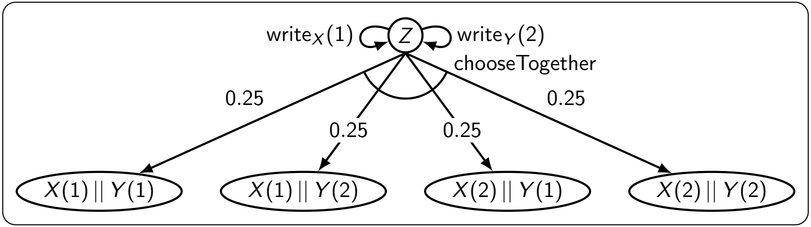

A linear format for prCRL: the LPPE

LPPEs are a subset of prCRL specifications:

$$X(\boldsymbol{g} : \boldsymbol{G}) = \sum_{\boldsymbol{d_1}:\boldsymbol{D_1}} c_1 \Rightarrow a_1 \sum\!\!\!\!\bullet_{\boldsymbol{e_1}:\boldsymbol{E_1}} f_1 : X(\boldsymbol{n_1}) \\ \ldots \\ + \sum_{\boldsymbol{d_k}:\boldsymbol{D_k}} c_k \Rightarrow a_k \sum\!\!\!\!\bullet_{\boldsymbol{e_k}:\boldsymbol{E_k}} f_k : X(\boldsymbol{n_k})$$

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Advantages of using LPPEs instead of prCRL specifications:

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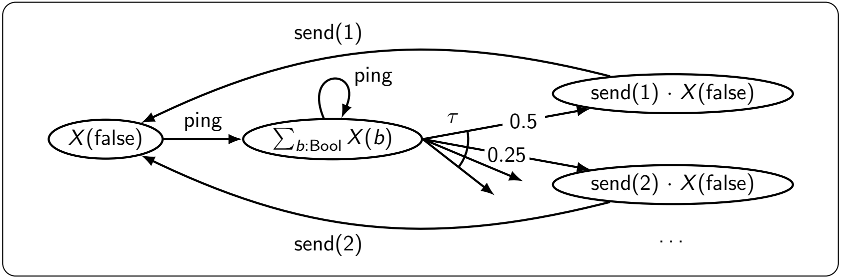
Theorem

Every specification (without unguarded recursion) can be linearised to an LPPE, preserving strong probabilistic bisimulation.

Linear Probabilistic Process Equations – an example



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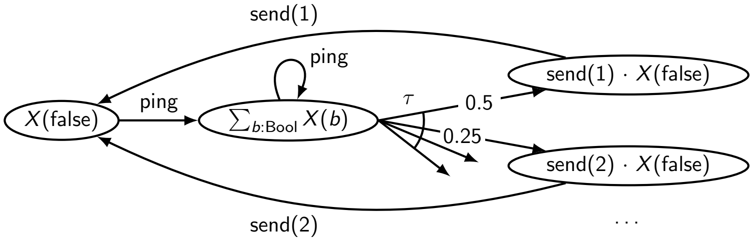

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Specification in LPPE

$$X(pc : \{1..3\}, n : \mathbb{N}^{\geq 0}) =$$
$$+ \; pc = 1 \Rightarrow \text{ping} \cdot X(2, 1)$$
$$+ \; pc = 2 \Rightarrow \text{ping} \cdot X(2, 1)$$
$$+ \; pc = 2 \Rightarrow \tau \sum_{n:\mathbb{N}^{>0}} \tfrac{1}{2^n} : X(3, n)$$
$$+ \; pc = 3 \Rightarrow \text{send}(n) \cdot X(1, 1)$$

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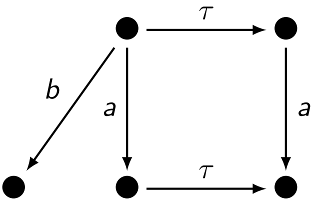
Branching bisimulation preservation by τ-steps

Unobservable τ-steps might disable behaviour...



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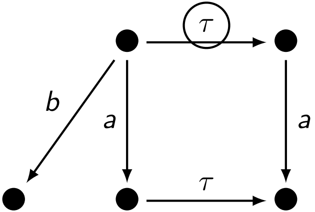

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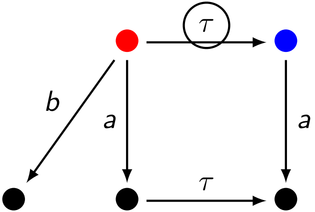

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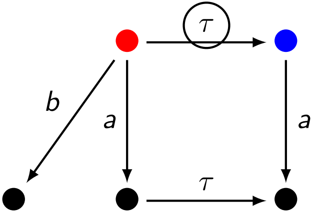

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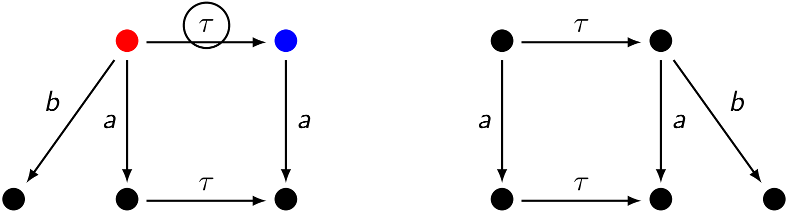

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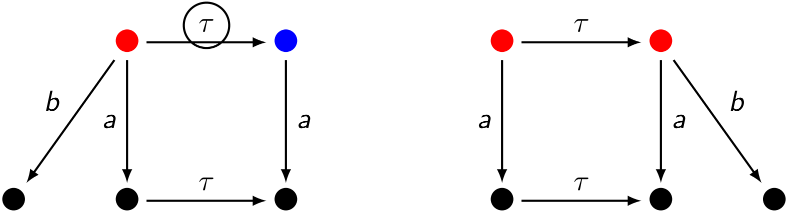

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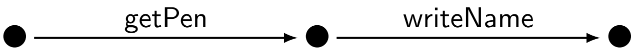

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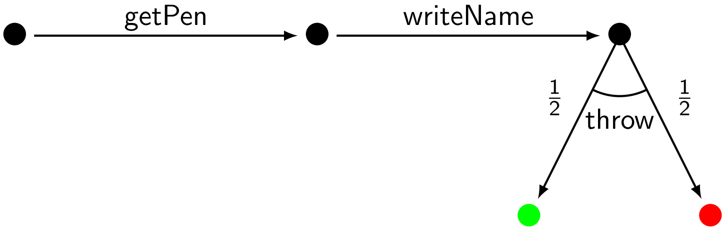

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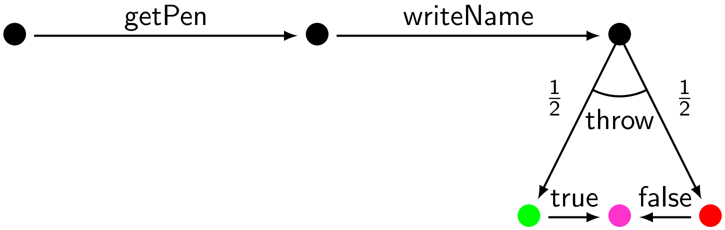

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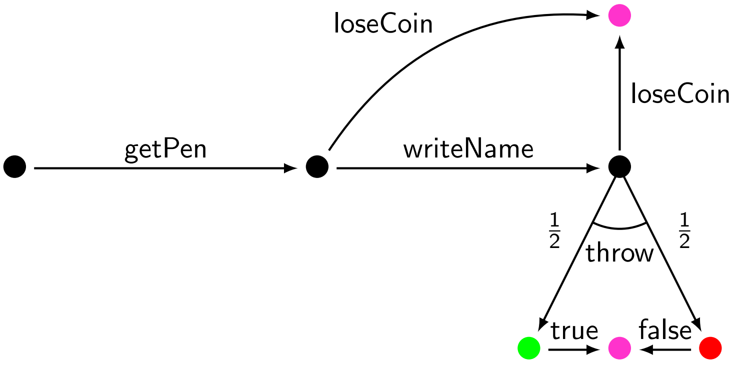

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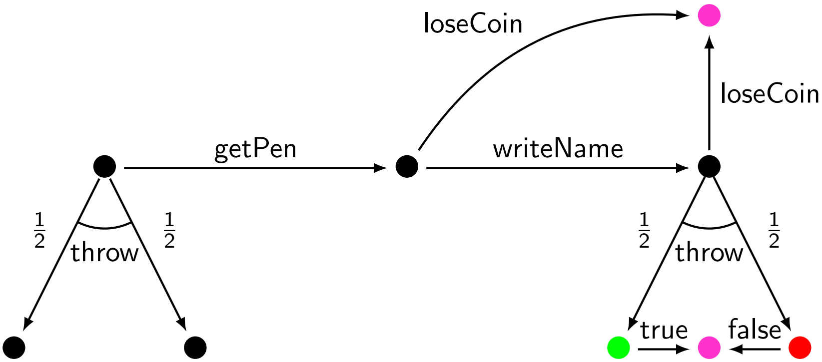

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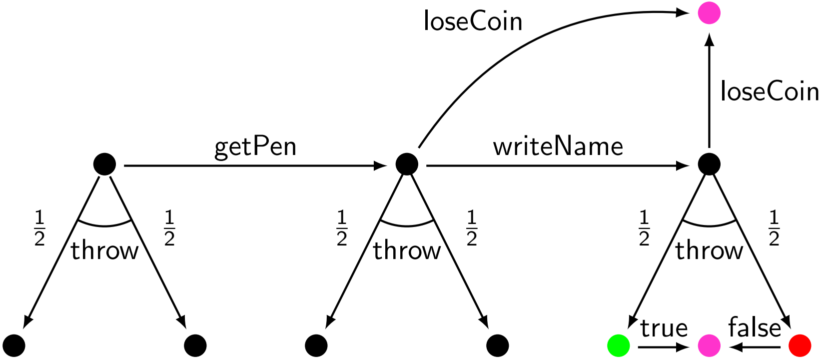

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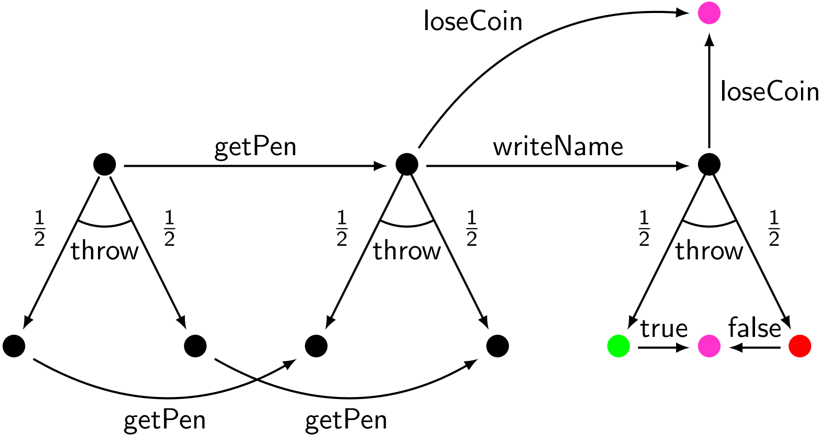

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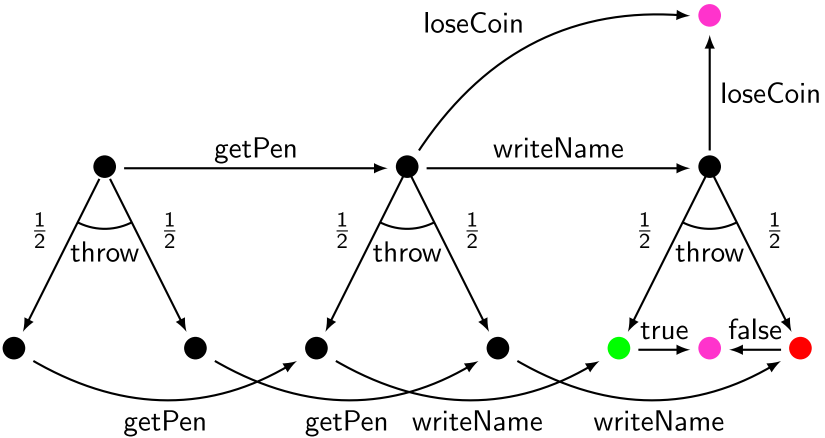

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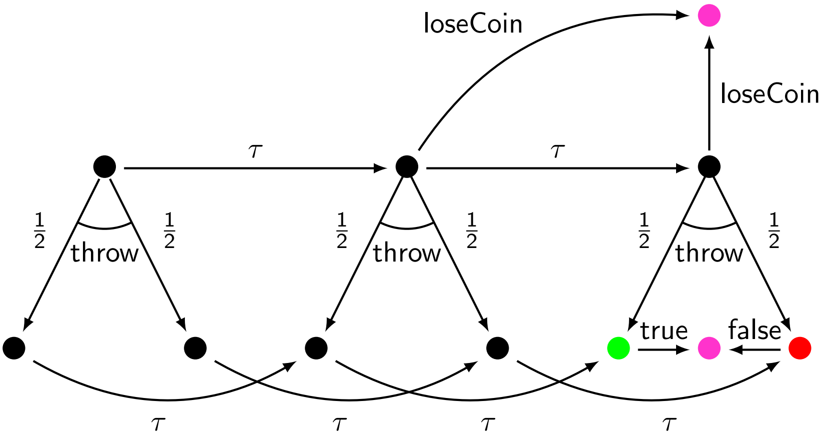

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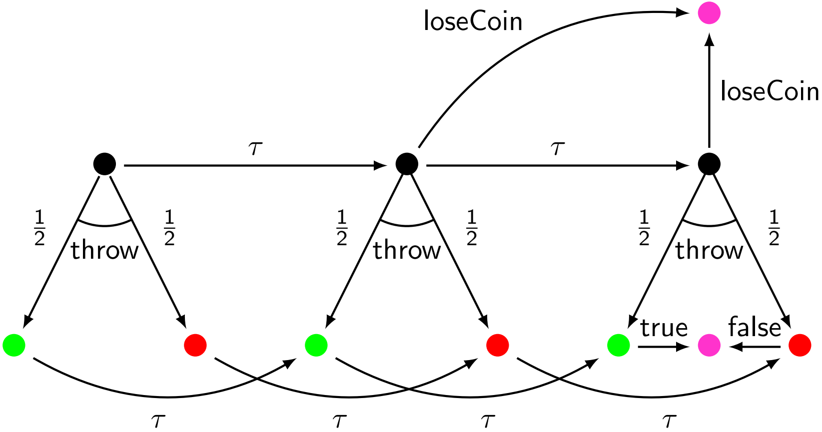

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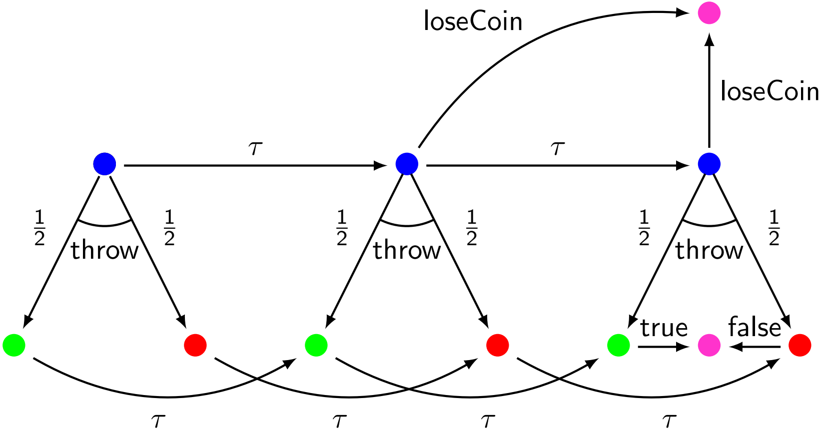

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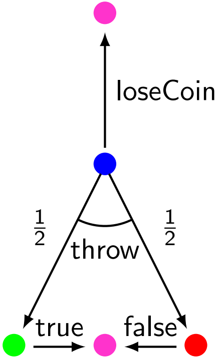

Confluence: non-probabilistic versus probabilistic

Three notions of confluence:

- weak confluence
- confluence
- strong confluence

Confluence: non-probabilistic versus probabilistic

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• weak confluence		• weak probabilistic confluence
• confluence	$\Rightarrow$	• probabilistic confluence
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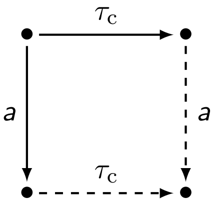

Strong confluence

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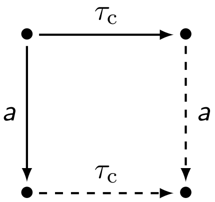

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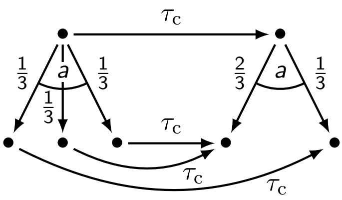

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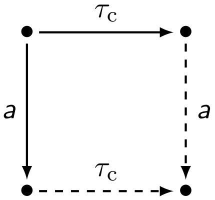

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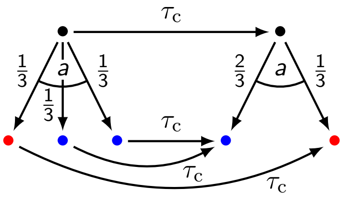

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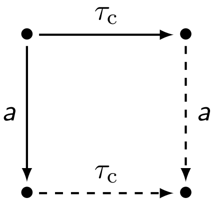

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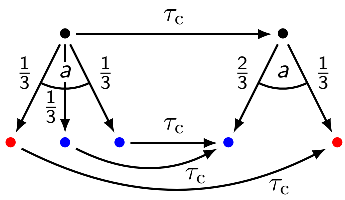
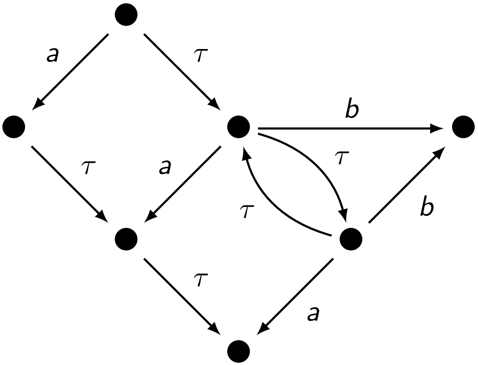


Strong probabilistic confluence

Theorem

States that are connected by confluent τ-steps are branching bisimilar.

State space reduction using confluence



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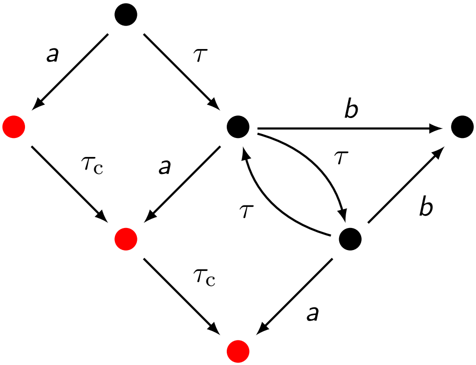

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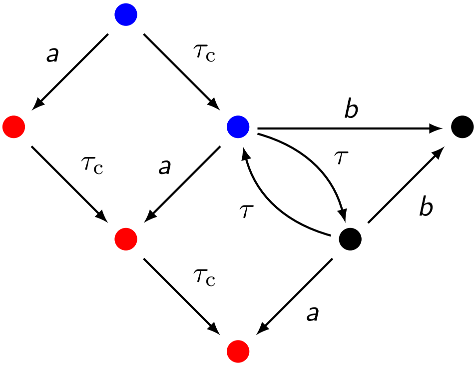

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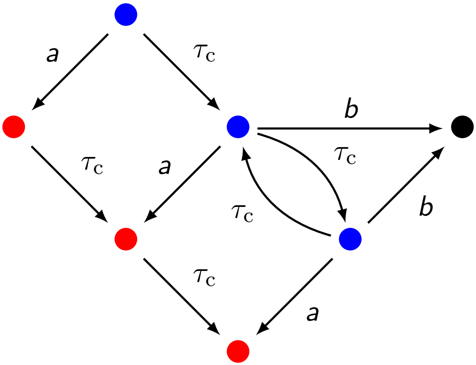

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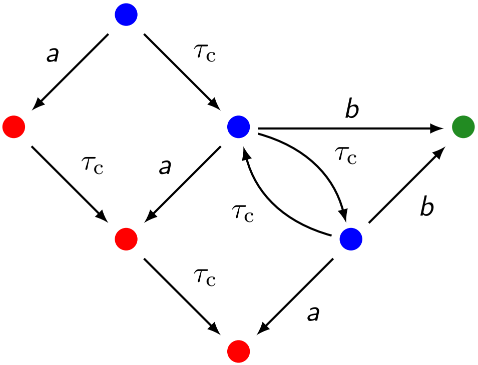

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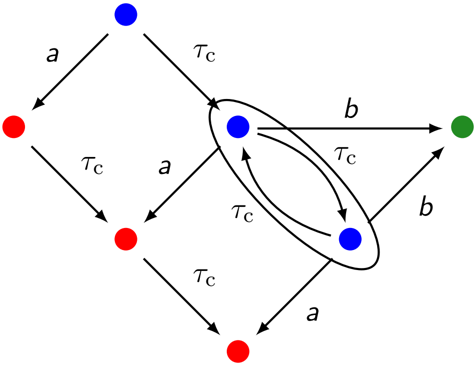

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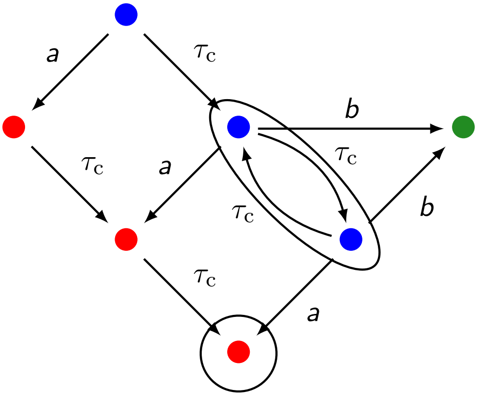

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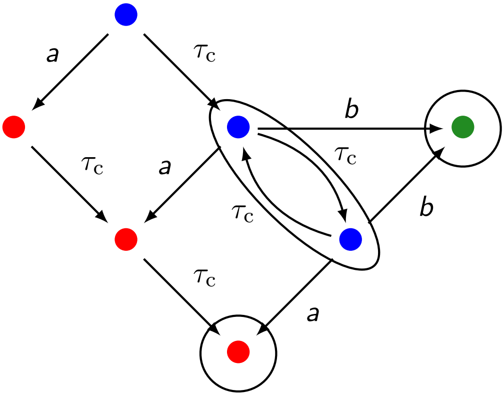

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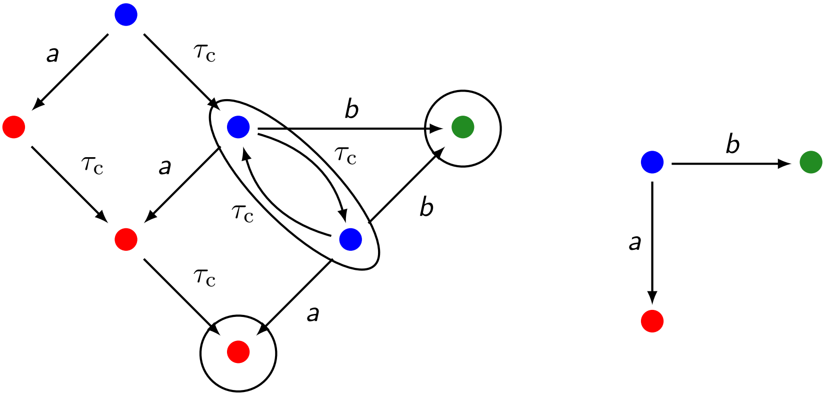

Table of Contents

Detecting confluence symbolically: LPPEs

Example specification

$$
\begin{aligned}
X(\mathrm{pc} : \{1..2\}, \mathrm{active} : \mathrm{Bool}) = \\
\sum_{n:\{1,2,3\}} \quad \mathrm{pc} = 1 \quad & \Rightarrow \mathrm{output}(n) \sum\!\!\!\bullet_{\mathrm{b:Bool}} \tfrac{1}{2} : X(2, \mathrm{b}) \\
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Detecting confluence symbolically: LPPEs

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How to know whether a summand is confluent?

- Its action should be τ
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- It should commute with all the other summands

Symbolic detection of confluence

$$X(\boldsymbol{g} : \boldsymbol{G}) = \sum_{\boldsymbol{d_i}:\boldsymbol{D_i}} c_i \Rightarrow a_i \sum_{\boldsymbol{e_i}:\boldsymbol{E_i}}\!\!\!\bullet\; f_i : X(\boldsymbol{n_i})$$
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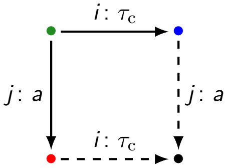

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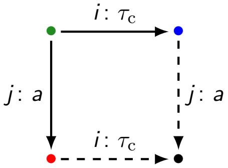

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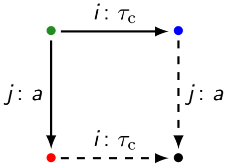

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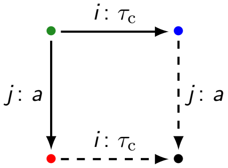

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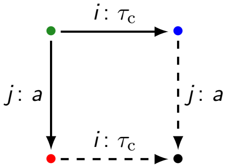

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$$\left(\qquad \right)$$

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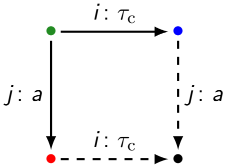

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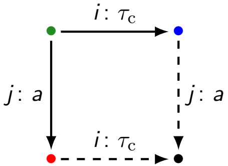

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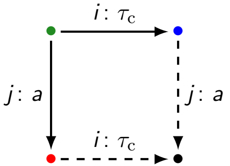

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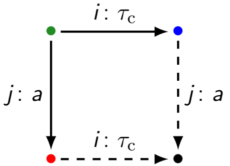

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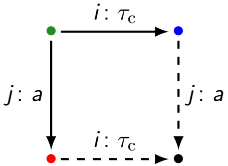

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Heuristics for detecting confluence

Heuristics for verifying the previous formula for summands i, j:

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More advanced leader election protocol

- Several processes each throw a die
- They asynchronously communicate the results
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- In case of a tie: continue with those processes

Applying confluence to the protocols

Specification	Original		Reduced		Runtime (sec)	
	States	Trans.	States	Trans.	Before	After
basicOriginal	3,763	6,158	631	758	0.45	0.22
basicReduced	1,693	2,438	541	638	0.22	0.13
leader-3-12	161,803	268,515	35,485	41,829	67.37	31.53
leader-3-15	311,536	515,328	68,926	80,838	145.17	65.82
leader-3-18	533,170	880,023	118,675	138,720	277.08	122.59
leader-3-21	840,799	1,385,604	187,972	219,201	817.67	211.87
leader-3-24	1,248,517	2,055,075	280,057	326,007	1069.71	333.32
leader-3-27	out of memory		398,170	462,864	–	503.85
leader-4-5	443,840	939,264	61,920	92,304	206.56	75.66
leader-4-6	894,299	1,880,800	127,579	188,044	429.87	155.96
leader-4-7	1,622,682	3,397,104	235,310	344,040	1658.38	294.09
leader-4-8	out of memory		400,125	581,468	–	653.60
leader-5-2	208,632	561,630	14,978	29,420	125.78	30.14
leader-5-3	1,390,970	3,645,135	112,559	208,170	1504.33	213.85
leader-5-4	out of memory		472,535	847,620	–	7171.73

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Number of states: -85% Number of transitions: -90%

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- We developed the process algebra prCRL, incorporating both data and probability, including a normal form (the LPPE) as starting point for symbolic optimisations
- We developed three new notions of confluence for PAs that preserve branching probabilistic bisimulation
- We showed how these notions can be used for state space reduction (even in the presence of τ-loops)
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M. Timmer, M.I.A. Stoelinga, and J.C. van de Pol.
Confluence reduction for probabilistic systems.
In *Proceedings of the 17th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2011)*.

Questions

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